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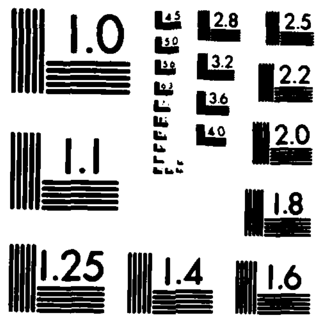

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A CLASS OF BIHARMONIC END-STRIP PROBLEMS ARISING IN ELASTICITY AND STOKES FLOW

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ABSTRACT

We consider boundary value problems for the biharmonic equation in the open rectangle $x > 0$, $-1 < y < 1$, with homogeneous boundary conditions on the free edges $y = \pm 1$, and data on the end $x = 0$ of a type arising both in elasticity and in Stokes flow of a viscous fluid, in which either two stresses or two displacements are prescribed. For such "non-canonical" data, coefficients in the eigenfunction expansion can be found only from the solution of infinite sets of linear equations, for which a variety of methods of formulation have been proposed.

A drawback of existing methods has been that the resulting equations are unstable with respect to the order of truncation. It is clear from an examination of the spectrum of a typical matrix that ill-conditioning is to be expected. However, a search among a wider class of possible trial functions than hitherto for use in a Galerkin method based on the actual eigenfunctions has led to the choice of a unique set, here termed optimal weighting functions, for which the resulting infinite matrix is diagonally-dominated. This ensures the existence of an inverse, which can be approximated by solving a finite subset of the equations.

Computations for a number of representative cases, presented in full in an internal report (Spence 1978) are summarized here, with emphasis on the rates of decay of the coefficients $\{c_m\}$ in the eigenfunction expansion. Knowledge of these decay rates is essential for a discussion of convergence, parallel to that given by Joseph (1977a,b) and his co-workers for canonical problems.

Asymptotic estimates of the decay rates have also been obtained by use of the solution of the biharmonic equation in a quarter plane. It is found that (i) for smooth continuous data satisfying compatibility conditions at the corners, the decay rates guarantee pointwise convergence. Also examined are (ii) cases of data violating compatibility (iii) discontinuous data and (iv) discontinuities in derivatives of the data. In these cases sharp estimates of convergence rates are obtained, which guarantee that integrals of the series converge to integrals of the data. The computations show striking confirmation of the theoretical estimates.

AMS (MOS) Subject Classifications: 31A30, 65F35, 73C35, 76D99

Key Words: Biharmonic, Eigenfunction expansion, Elasticity, Stokes flow, Galerkin, Optimal weighting functions, Asymptotics

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A CLASS OF BIHARMONIC END-STRIP PROBLEMS ARISING
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D. A. Spence*

1. Introduction

The solution of the biharmonic equation

$$\Delta^2 \Psi = 0 \tag{1.1}$$

in a semi-infinite strip Ω: $\{ x \geq 0, |y| \leq 1\}$ with homogeneous boundary conditions on the edges $y = \pm 1$ can be expressed formally as

$$\Psi(x,y) = \sum c_n \psi_n(y) e^{-\lambda_n x} \tag{1.2}$$

where $\psi_n(y) \equiv \psi(y,\lambda_n)$ are eigenfunctions associated with the names of Papkovitch (1940) and Fad'le (1941), λ_n are eigenvalues, and $\{c_n\}$ are coefficients that depend on the data on the edge Γ: $\{x = 0, |y| \leq 1\}$. In the case of solutions even in y, and such that $\Psi \to 0$ as $x \to \infty$, with edge conditions

$$\Psi = 0, \Psi_y = 0 \qquad \text{on } y = \pm 1, \tag{1.3}$$

the eigenvalues are the zeros with positive real parts of the function

$$C(\lambda) \equiv \lambda + \sin\lambda \cos\lambda \tag{1.4}$$

and the eigenfunctions** may be taken as

$$\begin{aligned} \psi(y,\lambda) &= (\lambda \cos^2\lambda)^{-1}(\sin\lambda \cos\lambda y - y \cos\lambda \sin\lambda y) \\ &\equiv (\lambda \cos^2\lambda)^{-1} \phi(y,\lambda) \qquad \text{say} \end{aligned} \tag{1.5}$$

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**These satisfy the equation $(D^2+\lambda^2)^2\psi = 0$ $(D \equiv \frac{d}{dy})$ with boundary conditions $\psi(1) = D\psi(1) = 0$, $D\psi(0) = D^3\psi(0) = 0$. They are also discussed by Lure (1964) Buchwald and Doran (1964) and many other authors in addition to those cited.

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The present paper is concerned only with data that can be expanded in terms of these even eigenfunctions. However there is also a set of eigenfunctions that are odd with respect to y, the eigenvalues in that case being zeros of

$$\hat{C}(\lambda) \equiv \lambda - \sin\lambda \cos\lambda \tag{1.6}$$

and the eigenfunctions multiples of

$$\hat{\phi} \equiv \cos\lambda \sin\lambda y - y \sin\lambda \cos \lambda y . \tag{1.7}$$

In general, an expansion such as (1.2) would involve eigenfunctions of both sets.

Footnote: Different authors have used different normalising factors from the present $(\lambda\cos^2\lambda)^{-1}$ when defining coefficients in the expansion (1.2) in terms of the basic eigenfunctions ϕ. Smith (1952), who is followed by Joseph (1977 et seq) writes the expansion as $\Psi = \sum\left(\frac{c_n}{\lambda_n}\right) e^{-\lambda_n x} \phi(y,\lambda_n)$, while Gregory (1980 a,b) writes $\Psi = \sum (-2)c_n e^{\frac{1}{2}\lambda_n x} \phi(y,\frac{1}{2}\lambda_n)$ (so that his λ_n's are twice those defined by (1.4), i.e. they are the roots of $\lambda + \sin\lambda = 0$). Thus the coefficients c_n obtained in the present paper are $\cos^2\lambda_n$ times those of Joseph, and $-\frac{1}{2}\lambda_n \cos^2\lambda_n$ times those of Gregory. For $n \gg 1$, these ratios are $O(n)$ and $O(n^2)$ respectively.

The reason for the present choice of normalizing factor is that it leads to the particularly simple biorthogonality relations (2.9), (2.10).

$C(\lambda)$ has a simple zero at $\lambda = 0$, and in addition a conjugate pair of zeros λ_n, $\bar{\lambda}_n$ in each interval $(n-\frac{1}{2})\pi < \mathrm{Re}\lambda < (n-1/4)\pi$. $-\lambda_n$, $-\bar{\lambda}_n$ are also zeros. Since we are concerned in this paper only with those zeros with positive real parts, we shall adopt the numbering convention that λ_1, λ_2 ..... are the zeros in the first quadrant, and

$$\lambda_{-n} = \bar{\lambda}_n, \qquad n = 1, 2, 3, \ldots \quad . \tag{1.8}$$

The sum (1.2) is then to be understood as

$$\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} c_n \, \psi_n(y) \, e^{-\lambda_n x} \equiv 2 \, \mathrm{Re} \sum_{n=1}^{\infty} c_n \, \psi_n e^{-\lambda_n x}, \tag{1.9}$$

the zero eigenfunction being excluded.

The asymptotic location of the roots is given by

$$\lambda_n = (n - 1/4)\pi + \frac{i}{2} \ln 4n\pi + o\,(1) \tag{1.10}$$

This expression shows that as $n \to \infty$

$$\left.\begin{aligned} & |\lambda_n| \sim n\pi, \quad |\cos\lambda_n|, \ |\sin\lambda_n| \sim (n\pi)^{\frac{1}{2}} \\ \text{while} \quad & \tan \lambda_n = i\,(1 - 1/4\lambda_n{}^2)^{\frac{1}{2}} \to i. \\ \text{Likewise} \quad & \tan \lambda_{-n} \to -i \end{aligned}\right\} \tag{1.11}$$

The first 10 eigenvalues, together with a much closer asymptotic expression, are listed in Table 1.

The zero eigenfunction satisfying the edge conditions in the form $\Psi = \Psi_y = 0$ is identically zero. However if the edge conditions are taken as $\Psi_{xx} = 0$, $\Psi_{xy} = 0$, which correspond in elastostatics to vanishing tractions, a term $\frac{1}{2}c_0 \, y^2$ corresponding to λ_0 could be included in (1.9).

Table 1. First 10 zeros of $2\lambda + \sin 2\lambda$

n	Re λ_n	Imλ_n
1	2.10619617	1.12536430
2	5.35626888	1.55157435
3	8.53668213	1.77554369
4	11.69917774	1.92940450
5	14.85406017	2.04685259
6	18.00493240	2.14189076
7	21.15341377	2.22172284
8	24.30034256	2.29055238
9	27.44620323	2.35104823
10	30.59129524	2.40501261

The asymptotic expression $\frac{1}{4}\ \zeta_n - \frac{\ln \zeta_n}{\zeta_n}$ gives 6 figure agreement with Re λ_n at $n = 4$, and $\frac{1}{2} \ln \zeta_n + [(\ln \zeta_n)^2 - 2\ln \zeta_n - 1]/\zeta_n^2$ agrees with Im λ_n to 4×10^{-5} at $n = 9$ ($\zeta_n = (4n - 1)\pi$).

2 Boundary value problems

The determination of the coefficients $\{c_n\}$ in the expansion (1.2) requires knowledge of two sets of boundary data on the edge Γ: $x = 0$, $|y| \leq 1$. For this purpose following a numerical classification scheme of the type introduced by Johnson & Little to treat a range of mixed boundary value problems we may define the following quantities:

$$\left.\begin{array}{ll} \Psi_{xy}(0,y) = f^{(1)}(y), & \Psi_{yy}(0,y) = f^{(2)}(y) \\ Q\ (0,y) = f^{(3)}(y), & P(0,y) = f^{(4)}(y) \end{array}\right\} \quad (2.1)$$

Here P (x,y) is the Laplacian $\Delta\Psi$ and Q the harmonic conjugate function defined so that $P + iQ$ is analytic in $x + iy$, with $Q(0,0) = 0$. An alternative notation used by Smith (1952) and followed by Joseph (1977) and Gregory (1980) is to write

$$\Psi_{xx}\ (0,y) = f(y), \quad \Psi_{yy}(0,y) = g(y) \quad (2.2)$$

To this it would be convenient to add

$$\Psi_{xy} = h, \qquad P = p,\ Q = q \quad \text{on } x = 0 \quad (2.3)$$

Then $f \equiv f^{(4)} - f^{(2)}, \quad g = f^{(2)}, \quad h = f^{(1)}, \quad p = f^{(4)}, \quad q = f^{(3)}.$ (2.4)

Formally, the derived functions possess expansions of the form

$$\begin{pmatrix} \Psi_{xy} \\ \Psi_{yy} \\ Q \\ P \end{pmatrix} = \sum c_n \begin{pmatrix} -\lambda_n \psi'_n \\ \psi''_n \\ \lambda_n^{-1}(\psi''_n + \lambda_n^2 \psi_n)' \\ \psi''_n + \lambda_n^2 \psi \end{pmatrix} e^{-\lambda_n x} \tag{2.5}$$

and with x = 0, following the notation of the author's earlier paper, the expansions can be written

$$\begin{pmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \\ f^{(4)} \end{pmatrix} = \sum c_n \begin{pmatrix} -\lambda\psi' \\ \psi'' \\ 2\sin\lambda y/\cos\lambda \\ -2\cos\lambda y/\cos\lambda \end{pmatrix}_{\lambda=\lambda_n} \equiv \sum c_n \begin{pmatrix} \phi_n^{(1)} \\ \phi_n^{(2)} \\ \phi_n^{(3)} \\ \phi_n^{(4)} \end{pmatrix} \tag{2.6}$$

Alternatively one can write expressions for f, g and h as

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \sum c_n \begin{pmatrix} \lambda^2\psi \\ \psi'' \\ -\lambda\psi' \end{pmatrix}_{\lambda=\lambda_n} \equiv \sum c_n \begin{pmatrix} \phi_n^{(4)} - \phi_n^{(2)} \\ \phi_n^{(2)} \\ \phi_n^{(1)} \end{pmatrix} \tag{2.7}$$

while $$\begin{pmatrix} p \\ q \end{pmatrix} = \sum 2(c_n/\cos\lambda_n) \begin{pmatrix} -\cos\lambda_n y \\ \sin\lambda_n y \end{pmatrix} \equiv \sum c_n \begin{pmatrix} \phi_n^{(4)} \\ \phi_n^{(3)} \end{pmatrix} \tag{2.8}$$

Canonical problems

The determination of the constants $\{c_n\}$ in the expansion (1.2) requires knowledge of two of the functions $f^{(\alpha)}$ (or, more generally, of two linear combinations of the $f^{(\alpha)}$). In general the $\{c_n\}$ must then be found from an infinite set of linear equations, but there are two problems for which they can be found explicitly, by use of certain biorthogonality relationships among the $\{\phi_n^{(\alpha)}\}$, namely

$$\tfrac{1}{2}\left\langle \phi_m^{(3)}\,\phi_n^{(1)} + \phi_m^{(1)}\phi_n^{(3)} \right\rangle = \delta_{mn} \tag{2.9}$$

and
$$\tfrac{1}{2}\left\langle \phi_m^{(4)}\,\phi_n^{(2)} + (\phi_m^{(2)}-\phi_m^{(4)})\phi_n^{(4)} \right\rangle = \delta_{mn} \tag{2.10}$$

Here $\langle\cdot\rangle$ denotes $\int_0^{1} (\cdot)\, dy$.

Thus (1) for the problem in which the prescribed data is the boundary values of $f^{(1)}$ and $f^{(3)}$, the scalar product of (2.6) with $(\phi_m^{(3)}, 0, \phi_m^{(1)}, 0)$ gives

$$c_m = \tfrac{1}{2}\left\langle \phi_m^{(3)} f^{(1)} + \phi_m^{(1)} f^{(3)} \right\rangle \tag{2.11}$$

In the context of plane elasticity/ as noted in Appendix A, with Ψ as the Airy stress function, this corresponds to a case in which the shear stress $-\Psi_{xy}$ and the normal displacement u are prescribed over Γ.

(2) Likewise the product with $(0, \phi_m^{(4)}, 0, \phi_m^{(2)} - \phi_m^{(4)})$ gives

$$c_m = \tfrac{1}{2}\left\langle \phi_m^{(4)} f^{(2)} + (\phi_m^{(2)} - \phi_m^{(4)})\, f^{(4)} \right\rangle \tag{2.12}$$

In the same way, this corresponds to the normal stress Ψ_{yy} and shear displacement v being prescribed over Γ.

In terms of the classification (2.7-8), the last two expressions are

$$c_n = (1/2) \left\langle (2\sin\lambda y/\cos\lambda) \quad h - (\lambda\psi')q \right\rangle_{\lambda=\lambda_n} \tag{2.13}$$

and

$$c_n = (1/2) \left\langle (-2\cos\lambda y/\cos\lambda) \quad g - (\lambda^2\psi)p \right\rangle_{\lambda=\lambda_n} \tag{2.14a}$$

$$= (1/2) \left\langle -\lambda^2\psi f + \psi'' g \right\rangle_{\lambda=\lambda_n} \tag{2.14b}$$

respectively.

3 Mixed problems

Three non-canonical problems of fundamental importance can be identified, as noted in Appendix A.

1. The elastic end-stress problem: Ψ_{xy}, Ψ_{yy} prescribed on Γ
2. The elastic end-displacement problem: $\Psi_{xy} + (1-\nu)Q$, $\Psi_{yy} - (1-\nu)P$, (ν = Poisson's ratio) prescribed.
3. The end-stress problem in Stokes flow : $\Psi_{xy} + \frac{1}{2}Q$, $\Psi_{yy} - \frac{1}{2}P$ prescribed on Γ

All three of these problems fall into the same framework if the data is prescribed in the form of values of the two functions

$$g^{(1)} = f^{(1)} + \alpha\, f^{(3)}, \qquad g^{(2)} = f^{(2)} - \alpha f^{(4)} \tag{3.1}$$

with $\alpha = 0,\ 1 - \nu,\ \frac{1}{2}$ in cases 1, 2, 3 respectively.
For these problems, the biorthogonality relations cannot be used to provide explicit expressions for the $\{c_n\}$, and it is necessary to resort to the formulation of an infinite set of linear equations from which $\{c_n\}$ can be found by truncation.

For instance in the case $\alpha = 0$, the $\{c_n\}$ must be such that the equations

$$f^{(1)} = \sum c_n\, \phi_n^{(1)} \tag{3.2a}$$

$$f^{(2)} = \sum c_n\, \phi_n^{(2)} \tag{3.2b}$$

are simultaneously satisfied pointwise on $0 \leq y \leq 1$. This will only be possible for sufficiently smooth data $(f^{(1)}, f^{(2)})$, and in particular

if the conditions

$$f^{(1)}(0) = 0,\ f^{(1)}(1) = 0,\ \int_0^1 f^{(2)}(y)dy = 0 \tag{3.3}$$

are satisfied (since each of the $\phi_n^{(1)}, \phi_n^{(2)}$ satisfy the corresponding conditions). Comparison with the known results for the canonical problems suggests that a sufficient condition, for data satisfying (3.3), is that $(f^{(1)}, f^{(2)}$ belongs to the Sobolev space $H^2(0,1)$, i.e. that $(f^{(1)''}, f^{(2)''}) \varepsilon L_2 (0,1)$. However a meaning can be assigned to the distribution of c_n obtainable from (3.2) under considerably more general conditions.

Formulation of infinite sets of equations for the $\{c_n\}$

Three essentially different methods for formulating equations occur in the literature:

(A) Direct collocation: In which a truncated set of the equations (3.2 a,b) are identically satisfied at a suitable set of points $\{y_m\}$ of the interval (0,1).

(B) Galerkin methods Here, sets of weighting functions $W_m^{(1)}$, $W_m^{(2)}$ are introduced and mean equations

$$(W_m^{(1)}\ f^{(1)}) = \sum(W_m^{(1)}\ \phi_n^{(1)})c_n,\quad (W_m^{(2)}\ f^{(2)}) = \sum(W_m^{(2)}\phi_n^{(2)})c_n \qquad (3.4a,b)$$

are obtained and solved simultaneously. The simplest such equations are provided by the set

$$W_m^{(1)} = \sin m\pi x,\ W_m^{(2)} = \cos m\pi x.$$

These were proposed by Benthem (1963), and are equivalent to those used by Gaydon & Shepherd. With our notation the matrix elements are given by

$$\left\langle \sin m\pi y\ \phi_n^{(1)}\right\rangle = -\left(\frac{\lambda_n}{m\pi}\right), \left\langle \cos m\pi y\ \phi_n^{(2)}\right\rangle = (-1)^{m-1}\frac{2m\pi\lambda_n^2\tan\lambda_n}{(\lambda_n^2-m^2\pi^2)^2} \qquad (3.5)$$

(C) Use of biorthogonal functions

Johnson & Little in effect add equations (2.11) and (2.12) to obtain

$$2c_m = \left(\tfrac{1}{2}\right)\left\langle \phi_m^{(3)}f^{(1)} + \phi^{(4)}f^{(2)}\right\rangle + \left(\tfrac{1}{2}\right)\left\langle \phi_m^{(1)}f^{(3)} + (\phi_m^{(2)} - \phi_m^{(4)})f^{(4)}\right\rangle \qquad (3.6)$$

On the right hand side, $f^{(1)}$ and $f^{(2)}$ are known, while $f^{(3)}$ and $f^{(4)}$ are calculated as $\sum c_n\ \phi_n^{(3)}$, $\sum c_n\ \phi_n^{(4)}$. From known quadratures we find the resulting bracket as

$$\left.\begin{aligned}\frac{1}{2}\left\langle \phi_m^{(1)}\phi_n^{(3)} + (\phi_m^{(2)} - \phi_m^{(4)})\phi_n^{(4)}\right\rangle = \frac{-2\lambda_m(\lambda_m\tan\lambda_m - \lambda_n\tan\lambda_n)}{(\lambda_m-\lambda_n)^2(\lambda_m+\lambda_n)} \equiv F_{mn}(m\neq n)\\ \text{say, while its value is 1 for } m = n.\end{aligned}\right\} \qquad (3.7)$$

This gives the set of equations

$$\left.\begin{aligned} c_m &= \sum F_{mn} c_n + d_m \\ \text{where} \quad d_m &= \frac{1}{2} \left\langle \phi_m^{(3)} f^{(1)} + \phi_m^{(4)} f^{(2)} \right\rangle \end{aligned}\right\} \tag{3.8}$$

It is to be noted however that this set of equations can also be obtained directly by a Galerkin method from (3.4a and b), using the weights $W_m^{(1)} = \frac{1}{2}\phi_m^{(3)}$, $W_m^{(2)} = \frac{1}{2}\phi_m^{(4)}$ and adding.

For an infinite set of equations of the form (3.8), Kantorovich & Krylov (1958) show that the solution $\{c_n\}$ is stable in the sense of approaching a limit as the number N of equations become large if the L_∞ norm $\|F\|_\infty \equiv \sup_m \sum_n |F_{mn}|$ is $\leq 1-\varepsilon, \varepsilon>0$. However this condition is not satisfied by the coefficients (3.7); in fact the sum $\sum_n |F_{mn}|$ tends to infinity with m .

To show this the equations may be written

$$\left.\begin{aligned} c_m &= \sum_{n=1}^{\infty} (F_{mn} c_n + F^*_{mn} \bar{c}_n) + d_m \\ \text{where} \quad F^*_{mn} &= -2\lambda_m(\lambda_m \tan\lambda_m - \bar{\lambda}_n \tan\bar{\lambda}_n)/(\lambda_m - \bar{\lambda}_n)^2(\lambda_m + \bar{\lambda}_n) \quad . \end{aligned}\right\} \tag{3.9}$$

The summation now extends over λ_n in the first quadrant only, and we need only consider λ_m in the first quadrant since the conjugate equation holds for $\bar{\lambda}_m$. The asymptotic expressions (1.11) show that

$$\lambda_m, \lambda_n \sim m\pi, \text{ while } \tan\lambda_m, \tan\lambda_n \to i.$$

Therefore

$$|F_{mn}| \sim \frac{2}{\pi} \frac{m}{|m^2-n^2|}, \quad |F^*_{mn}| \sim \frac{2m}{\pi} (m-n)^{-2} \quad (m \neq n) \tag{3.10}$$

(For $m = n$, $F_{mn} = 0$, $F^*_{mn} \sim 2im\pi/(\ell n 4m\pi)^2$). The imaginary part of F^*_{mm} does not appear in the diagonal when the equations (3.8) are formulated in real and imaginary form. Then

$$\sum |F_{mn}| \sim \frac{4}{\pi} \ell n\ 2m,\ \sum |F^*_{mn}| > \frac{m}{\pi} \{1+\frac{1}{2^2}+..\} = \frac{m\pi}{3}. \qquad (3.11)$$

Similar behaviour is exhibited by the elements (3.5) of the matrix formed by sine and cosine weighting functions, and we might expect this to be reflected in instability of the solution of truncated sets of these equations. In fact this was shown in numerical experiments described in [1].

It is evident that the difficulties arise from the factor $(\lambda_m-\lambda_n)^2$ in the denominator of F_{mn}.

Some light is thrown on the behaviour of the inverse matrix $(I - F)^{-1}$ by the distribution of eigenvalues of $I - F$. For $N = 4$, i.e. the 8×8 matrix, the eigenvalues are

$I - F$ (Johnson-Little): 1 ± 1.3749, $1 \pm .8337$, $1 \pm .9901$, 1 ± 1.0005 .

Thus the last 2 eigenvalues are extremely close to zero, and the matrix ill-conditioned (det $\doteq 5 \times 10^{-6}$).

By contrast, for the matrix $I - G$ derived in the next section by use of optimal weighting functions, the eigenvalues are

$I - G$: $1 \pm .3920$, $1 \pm .0828$, $1 \pm .0060$, $1 \pm .0001$

and it appears that the eigenvalues rapidly approach 1 for larger matrices, with $\det(I - G)$ of order 1.

Optimal weighting functions

As noted in the last section, the matrices resulting from the Galerkin methods B and C are not diagonally dominated, and this fact is reflected in the computations described in [1], where it was found that the coefficients obtained from these methods varied with the order of truncation.

However, an extra degree of freedom not exploited by previous authors is available in the choice of weighting functions. In the hope of producing a diagonally-dominated matrix, consider weighting functions of the general form

$$\left.\begin{aligned} \chi_m^{(1)} &= A\,\phi_m^{(1)} + B\,\phi_m^{(3)} \\ \chi_m^{(2)} &= C\,\phi_m^{(2)} + D\,\phi_m^{(4)} \end{aligned}\right\} \tag{3.12}$$

This is a natural extension of the biorthogonal weighting functions (3.7), and contains sufficient disposable constants to ensure the suppression of the factor $(\lambda_m - \lambda_n)^2$ in the denominator of the matrix elements.

If we write

$$f^{(1)} + \alpha\, f^{(3)} = g^{(1)}, \quad f^{(2)} - \alpha\, f^{(4)} = g^{(2)} \tag{3.1 bis}$$

then the stress ($\alpha = 0$) and displacement ($\alpha = 1 - \nu$) problems may be treated together as cases in which the functions $g^{(1)}$, $g^{(2)}$ are known on $0 \le y \le 1$. We shall therefore seek optimal weighting functions in the above sense for $g^{(1)}$ and $g^{(2)}$, and expect the result to depend on the parameter α. Thus from the equations

$$\left.\begin{aligned} g^{(1)} &= \sum c_n\,(\phi_n^{(1)} + \alpha\phi_n^{(3)}) \\ g^{(2)} &= \sum c_n\,(\phi_n^{(2)} - \alpha\phi_n^{(4)}) \end{aligned}\right\} \tag{3.13}$$

we obtain the equations

$$\left.\begin{aligned} &\sum A_{mn} c_n = d_m \\ &\text{with } A_{mn} = \left\langle \chi_m^{(1)}\left(\phi_n^{(1)} + \alpha\phi_n^{(3)}\right) + \chi_m^{(2)}\left(\phi_n^{(2)} - \alpha\phi_n^{(4)}\right) \right\rangle \\ &\phantom{\text{with }} d_m = \left\langle \chi_m^{(1)} g^{(1)} + \chi_m^{(2)} g^{(2)} \right\rangle \end{aligned}\right\} \quad (3.14)$$

Quadratures of the eigenfunctions appendix of (1) show that for $m \neq n$

$$A_{mn} = \frac{4\lambda_m\lambda_n}{(\lambda_m^2-\lambda_n^2)} \left[-\frac{A(\lambda_m^2 + \lambda_n^2) + 2C\lambda_m\lambda_n}{\lambda_m^2 - \lambda_n^2} + B - \alpha A + \left(\frac{\lambda_m^2 D + \alpha\lambda_n^2 C}{\lambda_m\lambda_n}\right)\right] C_{mn}$$

$$+ 4\alpha\left[(B-D)\, C_{mn} - B\, T_{mn}\right]$$

$$\left.\begin{aligned} &\text{where } C_{mn} = \left\langle \frac{\cos\lambda_m y \cos\lambda_n y}{\cos\lambda_m \cos\lambda_n} \right\rangle = \left(\lambda_m \tan\lambda_m - \lambda_n \tan\lambda_n\right)/(\lambda_m^2 - \lambda_n^2) \\ &\text{and } T_{mn} = (\tan\lambda_m + \tan\lambda_n)\,/\,(\lambda_m + \lambda_n) \end{aligned}\right\} \quad (3.15)$$

while

$$A_{mm} = \tfrac{1}{2}A + \frac{3}{2}C + \frac{1}{3}(\lambda_m \tan\lambda_m)(A + C) + B + D$$

$$+ \alpha\left[A - C - 4B\left(\frac{\tan\lambda_m}{\lambda_m}\right)\right]$$

To suppress the factor $(\lambda_m - \lambda_n)^3$ in the denominator of A_{mn} it is necessary to set $A = -C$. The choice $C = 1$, $A = -1$ reduces the expression to

$$4\left[\frac{\lambda_m\lambda_n}{(\lambda_m+\lambda_n)^2} + \frac{(B+\alpha)\lambda_m\lambda_n + \lambda_m^2 D + \lambda_n^2\alpha}{\lambda_m^2 - \lambda_n^2} + \alpha(B-D)\right] C_{mn} - 4\alpha B T_{mn}$$

If now we set $B = \frac{1}{2} - \alpha$, $D = -\frac{1}{2} - \alpha$, the remaining factors of $(\lambda_m - \lambda_n)$ in the denominator are suppressed leaving

$$\left.\begin{aligned} A_{mn} &= -G_{mn} - 2\alpha(1-2\alpha)T_{mn} \qquad m \neq n \\ \text{say, where } G_{mn} &= 2\lambda_m(\lambda_m \tan\lambda_m - \lambda_n \tan\lambda_n)/(\lambda_m + \lambda_n)^3 \\ \text{while} \quad & \\ A_{mm} &= (1-4\alpha) - 2\alpha(1-2\alpha)T_{mm} \qquad \left(T_{mm} = \frac{\tan\lambda_m}{\lambda_m}\right) \end{aligned}\right\} \tag{3.16}$$

Hence if the weighting functions are chosen as

$$\left.\begin{aligned} \chi_m^{(1)} &= -\phi_m^{(1)} + (\tfrac{1}{2}-\alpha)\,\phi_m^{(3)} \\ \chi_m^{(2)} &= \phi_m^{(2)} - (\tfrac{1}{2}+\alpha)\,\phi_m^{(4)} \end{aligned}\right\} \tag{3.17}$$

the equations (3.14) are of the form

$$(1-4\alpha)\,c_m = \sum \{G_{mn} + 2\alpha(1-2\alpha)T_{mn}\}\,c_n + d_m \tag{3.18}$$

In particular, for the traction problem ($\alpha = 0$) they reduce to

$$c_m = \sum G_{mn}\,c_n + d_m \tag{3.19}$$

Likewise for the Stokes problem ($\alpha = \frac{1}{2}$) the system is

$$-c_m = \sum G_{mn}\,c_n + d_m \tag{3.20}$$

Diagonal dominance of the matrix $A \equiv I - G$

We now confirm that the infinite matrix I-G is strictly diagonally dominated, i.e. that

$$\sum_{n=-\infty}^{\infty} |G_{mn}| < 1 \quad \forall_m \tag{3.21}$$

An estimate of the absolute row sums of G_{mn} may be obtained by use of the asymptotic values

$$\lambda_m \doteq \text{Re}\,\lambda_m \sim (|m| - \tfrac{1}{4})\,\pi$$

$$\lambda_m \tan\lambda_m = \frac{1}{2} \pm (\frac{1}{4} - \lambda_m^2)^{\frac{1}{2}} \quad \sim -\tfrac{1}{2} \pm \; i\lambda_m \; (m \gtrless 0)$$

Hence in the numerator of $|G_{mn}|$,

$$|\lambda_m \tan\lambda_m - \lambda_n \tan\lambda_n| \sim |(m - \frac{1}{4}) \mp (n - \frac{1}{4})|\pi$$

accordingly as m, n have the same or opposite signs.

Likewise the denominator of $|G_{mn}|$ namely

$$|\lambda_m + \lambda_n|^3 \geq (\text{Re}\lambda_m + \text{Re}\lambda_n)^3 \sim \pi^2 \; (|m|+|n|- \tfrac{1}{2})^3$$

Hence for m, n > 0,

$$|G_{mn}|+|G_{m,-n}| \sim \left(\frac{2}{\pi}\right) (m- \frac{1}{4}) \left[\frac{|m-n| + (m+n-\frac{1}{2})}{(m+n-\frac{1}{2})^3}\right] \equiv g(m,n) \text{ say} \tag{3.22}$$

This leads to an estimate of the m^{th} absolute row sum of the truncated matrix of order 2N as

$$\sum_{n=-N}^{N} |G_{mn}| \sim \sum_{n=1}^{N} g(m,n) \leq \int_{\frac{1}{2}}^{N+\frac{1}{2}} g(m,n)\, dn \tag{3.23}$$

(the inequality holds since g is decreasing and concave as n increases).

If $m < N$, the integral equals $s_m - R_m^{(N)}$ say,

$$\text{with } S_m = \frac{1}{\pi}\left[2\left(1 - \frac{1}{4m}\right)^2 + 1\right], \quad R^{(N)} = \frac{2}{\pi}\left[1 - \left(\frac{N + \frac{1}{4}}{N + m}\right)^2\right] \tag{3.24}$$

The estimate for the infinite matrix is thus $\sum_{n=-\infty}^{\infty} |G_{mn}| \sim |S_m| \leq \frac{3}{\pi}$ (3.25)

For a large finite matrix, the estimate

$$\sum_{n=-N}^{N} |G_{mn}| \sim S_m - R_m^{(N)}$$

is well borne out by computed values of the absolute row sum, as shown in Table 2 for the cases N = 30 and N = 99.

The fact that $\sum_{n=-\infty}^{\infty} |G_{mn}| \leq \eta < 1$ is sufficient to ensure that for bounded $|d_m|$, the equations (3.19) possess a unique bounded inverse $\{c_m\}$. In principle this can be constructed from the sequence

$$\underset{\sim}{c} = (I + G + G^2 + \dots)\, \underset{\sim}{d} \tag{3.27}$$

A theorem of Kantorovitch & Krylov shows that the inverse $c_m^{(N)}$ to the truncated set of equations

$$c_m^{(N)} = \sum_{N=-N}^{N} G_{mn}\, c_n^{(N)} + d_m^{(N)} \tag{3.28}$$

converges to a unique limit $\{c_m\}$ as $N \to \infty$.

These remarks also apply to (3.20), the inverse in that case being

$$\underset{\sim}{c} = -(I - G + G^2 - \dots)\, \underset{\sim}{d} \tag{3.29}$$

In particular, in either case $\underset{\sim}{d} = 0 \Rightarrow \underset{\sim}{c} = 0$ (3.30)

Table 2

m	$\sum_{n=-30}^{30} \lvert G_{mn} \rvert$	$S_m - R_m^{(30)}$	$\sum_{n=-99}^{99} \lvert G_{mn} \rvert$	$S_m - R_m^{(99)}$
1	.63515	.64598	.65744	.66690
2	.74430	.73800	.79091	.78385
3	.75863	.75157	.82716	.81938
4	.75142	.74515	.83923	.83233
5	.73713	.73179	.84194	.83603
6	.72040	.71586	.84021	.83517
7	.70306	.69918	.83610	.83177
8	.68593	.68257	.83063	.82688
9	.66937	.66644	.82436	.82108
10	.65355	.65097	.81759	.81470
15	.58494	.58646	.78152	.77980
20	.53652	.53551	.74648	.74534
25	.49894	.49822	.71432	.71349
30	.47011	.46956	.68522	.68459
40			.63535	.63495
50			.59471	.59443
60			.56128	.56107
70			.53350	.53334
80			.51019	.51006
90			.49044	.49034
99			.47515	.47506

The first and second columns are the two sides of (3.26) for $N = 30$, the third and fourth for $N = 99$.

4. Asymptotic estimates of convergence rates

In this section asymptotic estimates will be made of the rate of convergence of the coefficients $\{c_n\}$ for the mixed problem in which $f^{(1)}$, $f^{(2)}$ are prescribed functions of y on the interval [0,1]. Four types of data will be considered

(i) Smooth continuous data satisfying the compatability condition $f^{(1)}(1) = 0$.

(ii) Data for which the above compatability condition is violated.

(iii) Data containing a discontinuity at an internal point $c \in (0,1)$.

(iv) Data containing a discontinuity in slope at an internal point c.

In each case it will be assumed that the data satisfies the additional requirements at y = 0 for expansion in terms of the even eigenfunctions. These are

$$f^{(1)}(0) = f^{(1)''}(0) = 0, \qquad f^{(2)'}(0) = f^{(2)'''}(0) = 0. \tag{4.1}$$

The technique in cases (i) and (ii) will be to estimate the behaviour of the functions $f^{(3)}$ and $f^{(4)}$, which is not known in advance, by solving the biharmonic equation in a quarter plane with boundary conditions on the two sides that are asymptotically the same as those in the neighbourhood of the corner $x = 0$, $y = -1$ in the strip problem. In cases (iii) and (iv), in which the governing singularity in the data is at an interior point, the relevant asymptotic solution of the biharmonic equation is in a half-plane. Having obtained estimates of $f^{(3)}$ and $f^{(4)}$, we can then estimate c_n from (2.11) or (2.12), using in addition the known values of $f^{(1)}$ or $f^{(2)}$.

Case (i) : Smooth data satisfying the compatability conditions

Since in this case $f^{(1)}(-1) = 0$, we may assume the data can be expanded near $y = -1$ in the form

$$f^{(1)}(y) \equiv h(Y) = -a_1 Y - a_2 Y^2 - \dots \tag{4.2a}$$

$$f^{(2)}(y) \equiv g(Y) = b_o + b_1 Y + b_2 Y^2 + \dots \tag{4.2b}$$

where $Y = 1 + y$.

The solution of the biharmonic equation in the quarter plane $X, Y \geq 0$, subject to the boundary conditions

$$\Psi = 0, \ \Psi_Y = 0 \qquad \text{on } Y = 0, \tag{4.3a}$$

$$\Psi_{XY} = h(Y), \ \Psi_{YY} = g(Y) \text{ on } X = 0 \tag{4.3b}$$

is found in Appendix C as a Mellin integral

$$\Psi = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{1-s} \ \hat{\Psi}(s,\theta) \, ds \tag{4.4}$$

with $-1 < c < 0$, $r = (X^2 + Y^2)^{\frac{1}{2}}$, $\theta = \tan^{-1} \frac{Y}{X}$, $\hat{\Psi}$ being expressed in terms of Mellin transforms of the data on the boundary $\theta = \frac{\pi}{2}$:

$$\begin{pmatrix} \hat{g}(s) \\ \hat{h}(s) \end{pmatrix} = \int_0^{\infty} \begin{pmatrix} g(Y) \\ h(Y) \end{pmatrix} Y^s dY \tag{4.5}$$

The values of the Laplacian $P = \Delta\Psi$ and its harmonic conjugate Q on the boundary $\theta = \pi/2$ are obtained as the inverse Mellin transforms

$$\begin{pmatrix} p(\Upsilon) \\ q(\Upsilon) \end{pmatrix} = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \begin{pmatrix} \hat{p}(s) \\ \hat{q}(s) \end{pmatrix} \Upsilon^{-s-1} ds \tag{4.6}$$

$$\left.\begin{aligned} &\text{where} \quad \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = -\left(\frac{2}{\Delta(s)}\right) M(s) \begin{pmatrix} \hat{g} \\ \hat{h} \end{pmatrix} \\ &\text{with} \quad M(s) = \begin{pmatrix} (s+\sin^2 \frac{\pi s}{2}) & \sin \frac{\pi s}{2} \cos \frac{\pi s}{2} \\ \sin \frac{\pi s}{2} \cos \frac{\pi s}{2} & (s - \sin^2 \frac{\pi s}{2}) \end{pmatrix} \\ &\text{and } \Delta = \det M = s^2 - \sin^2 \frac{\pi s}{2} \end{aligned}\right\} \tag{4.7}$$

Δ has real zeros at 0, 0, $\pm$ 1, and a doubly-infinite set of complex zeros $\pm \gamma_2$, $\pm \bar{\gamma}_2$, $\pm \gamma_4$, $\pm \bar{\gamma}_4$,... which are spaced so that $2n < \mathrm{Re}\gamma_{2n} < 2n + 1$

The first of these is

$$\gamma_2 = 2.739593 + i\,(1.119025) \tag{4.8}$$

The Mellin transforms of g and h must be of the form

$$\left.\begin{aligned} \hat{h}(s) &= -\frac{a_1}{s+2} - \frac{a_2}{s+3} + \text{function regular in } -3 \leq \mathrm{Re}\,s < 0 \\ \hat{g}(s) &= \frac{b_o}{s+1} + \frac{b_1}{s+2} + \frac{b_2}{s+3} + \text{function regular in } -3 \leq \mathrm{Re}\,s < 0 \end{aligned}\right\} \tag{4.9}$$

since on inversion in accordance with (4.5), these produce the stated expansions. Then from (4.7),

$$\hat{p}(s) = \left(-\frac{2}{\Delta}\right)\left(s+\sin^2 \frac{\pi s}{2}\right) \hat{g} - \left(\frac{\sin \pi s}{\Delta}\right) \hat{h} \tag{4.10}$$

On inversion using (4.6) we obtain

$$p(Y) = b_o + b_1 Y + C Y^{\gamma_2 - 1} + \bar{C}\, Y^{\bar{\gamma}_2 - 1} + O(Y^2) \tag{4.11}$$

where C is a (complex) constant.

Since $f^{(2)}$, $f^{(4)}$ are symmetrical with respect to y, they can now be expressed on the interval $0 \leq y \leq 1$ by writing 1-y for Y in (4.2b) and (4.11) respectively, as

$$f^{(2)} = b_o + b_1 (1-y) + O\,(1-y)^2 \tag{4.12a}$$

$$f^{(4)} = b_o + b_1 (1-y) + C\,(1-y)^{\gamma_2 - 1} + \bar{C}\,(1-y)^{\bar{\gamma}_2 - 1} + \ldots \tag{4.12b}$$

c_n can now be evaluated from (2.12) as

$$c_n = \frac{1}{2} \left\langle - \left(\frac{2\cos\lambda y}{\cos\lambda}\right) (f^{(2)} - f^{(4)}) + \psi'' f^{(4)} \right\rangle_{\lambda=\lambda_n}$$

Two integrations by parts give

$$c_n = \frac{1}{2\lambda_n^{\,2}} \left\langle \frac{2\cos\lambda_n y}{\cos\lambda_n} f^{(2)''} \right\rangle + \frac{1}{2} \left\langle \left(\psi_n - \frac{2\cos\lambda_n y}{\lambda_n^2 \cos\lambda_n}\right) f^{(4)''} \right\rangle \tag{4.13}$$

For $f^{(2)}$, $f^{(4)}$ ϵ H^2 (0,1), bounds on these terms are provided by the Cauchy-Schwarz inequality, since from [1] we have the L_2 norms

$$\left\| \frac{2\cos\lambda_n y}{\cos\lambda_n} \right\| \sim \text{const} \,/\, (\ln n)^{\frac{1}{2}} \tag{4.14a}$$

$$\| \psi_n \| \sim \text{const} \,/\, n\,(\ln n)^{3/2} \tag{4.14b}$$

At this stage therefore our best estimate is

$$|c_n| = O\left(1/n\,(\ln n)^{3/2}\right), \tag{4.15}$$

since $f^{(4)}$ cannot be expected to have stronger differentiability properties than the primary data.

However for smoother data such that $f^{(2)} \varepsilon\ H^4\ (0,1)$, we may write

$$\left.\begin{aligned} f^{(2)''} &= u(y) \\ f^{(4)''} &= K\ (1-y)^{\nu-1} + \bar{K}\ (1-y)^{\bar{\nu}-1} + v(y) \end{aligned}\right\} \tag{4.16}$$

where $K = (\gamma_2 - 1)(\gamma_2 - 2)C$, $\nu = \gamma_2 - 2$, and $u(y)$, $v(y)$ are such that u'', $v'' \ \varepsilon\ L_2\ (0,1)$

Then

$$c_n = \frac{1}{2}\left\langle\left(\frac{2\cos\lambda_n y}{\lambda_n^2\cos\lambda_n}\right)(u - v) + \psi_n\ v\right\rangle$$

$$+ \frac{1}{2}\left\langle\left(\psi_n - \frac{2\cos\lambda_n y}{\lambda_n^2\cos\lambda_n}\right)\left(K\ (1-y)^{\nu-1} + \bar{K}\ (1-y)^{\bar{\nu}-1}\right)\right\rangle \tag{4.17}$$

Further integration by parts shows that the first term is $O(n^{-3})$.

We now focus attention on the second bracket. The part multiplied by K when written in full is

$$(2\lambda\cos^2\lambda)^{-1}\left\langle\{\sin\lambda(1-y) + (1-y)\cos\lambda\sin\lambda y - \frac{2}{\lambda}\cos\lambda\cos\lambda y\}(1-y)^{\nu-1}\right\rangle_{\lambda=\lambda_n} \tag{4.18}$$

After integration of the middle term by parts, and on writing $y = 1-t$, this becomes

$$(2\lambda\cos^2\lambda)^{-1}\left\langle\{\sin\lambda t - (2+\nu)\frac{\cos\lambda}{\lambda}\cos\lambda\ (1-t)\}\ t^{\nu-1}\right\rangle + (2\lambda^2\cos\lambda)^{-1} \tag{4.19}$$

The integral can be estimated for large n, by use of asymptotic formulae derived in Appendix D for $\lambda\ \varepsilon\ \Lambda+$, i.e. the set of λ for which $C(\lambda) = 0$, $\mathrm{Re}\lambda > 0$, and $0 < \mathrm{Re}\nu < 1$:

$$\left\langle t^{\nu-1}\cos\lambda t\right\rangle = \lambda^{-\nu}\left(\cos\frac{\nu\pi}{2}\right)\Gamma(\nu) + \frac{\sin\lambda}{\lambda} + O(\lambda^{-3/2}) \tag{4.20a}$$

$$\left\langle t^{\nu-1}\sin\lambda t\right\rangle = \lambda^{-\nu}\left(\sin\frac{\nu\pi}{2}\right)\Gamma(\nu) - \frac{\cos\lambda}{\lambda} + O(\lambda^{-3/2}) \tag{4.20b}$$

When these are substituted in (4.19) the contributions from the second terms on the right cancel perfectly with the term $1/2\lambda^2 \cos\lambda$, and we are left with

$$\left(-\frac{1}{2}\right)\lambda^{-\nu-2}\,\Gamma(\nu)\left[(2+\nu)\cos\frac{\nu\pi}{2} + (3+\nu)\left(\sin\frac{\nu\pi}{2}\right)\tan\lambda\right] + O(\lambda^{-7/2})$$

Since $\nu = \gamma_2 - 2$, where γ_2 is such that $\gamma_2 + \sin\frac{\pi\gamma_2}{2} = 0$,

this expression can be written in the form

$$\left.\begin{aligned} &\lambda^{-\gamma_2}(A + B\tan\lambda) + O(\lambda^{-7/2}) \\ \text{with}\quad &\frac{B}{A} = \frac{\gamma_2 - 1}{\cos\frac{\pi\gamma_2}{2}} \end{aligned}\right\} \qquad (4.21)$$

Then since $\lambda_n \sim n\pi + O\left(\frac{\ln n}{n}\right)$, while $\tan\lambda_n \sim \pm i$, the estimate of $|c_n|$ is

$$|c_n| \sim (\text{const})\, n^{-2.74} + O(n^{-3}) \qquad (4.22)$$

This estimate is well supported by computations using the method of optimal weighting functions, for two cases for which the coefficients c_n were tabulated in [1]. In the nomenclature of that report, they are

$$\begin{aligned} &\text{Case 1:}\quad f^{(1)} = 0,\ f^{(2)} = 1-3y^2 \quad \left[d_m = -2\left(\frac{\tan\lambda_m}{\lambda_m}\right)\left(1 - \frac{9}{\lambda_m^2}\right) - \frac{6}{\lambda_m^2}\right] \\ &\text{Case 3:}\quad f^{(1)} = y - y^3,\ f^{(2)} = 0 \quad \left[d_m = -6\left(\frac{\tan\lambda_m}{\lambda_m^2}\right)\left(1 - \frac{5}{\lambda_m^2}\right) - \frac{18}{\lambda_m^3}\right] \end{aligned} \qquad (4.23)$$

The values of $n^{2.74} c_n$, with $N = 40$, are listed in Table 3, and it is seen that in both cases, they vary only slowly with n, the maximum variation between n = 12 and n = 40 being of order ± 8 per cent.

Table 3 <u>Decay of coefficients for the smooth continuous data listed in (4.23)</u>

	Case 1	Case 3		Case 1	Case 3
n	$n^{2.74}\lvert c_n\rvert$	$n^{2.74}\lvert c_n\rvert$	n	$n^{2.74}\lvert c_n\rvert$	$n^{2.74}\lvert c_n\rvert$
1	.8676	.3704	21	.4555	.4854
2	.4735	.6337	22	.4566	.4945
3	.6175	.6465	23	.4586	.5030
4	.6774	.6012	24	.4615	.5108
5	.6912	.5473	25	.4656	.5179
6	.6820	.4978	26	.4697	.5243
7	.6616	.4586	27	.4742	.5301
8	.6362	.4291	28	.4790	.5352
9	.6093	.4096	29	.4839	.5398
10	.5830	.3988	30	.4886	.5438
11	.5583	.3951	31	.4940	.5473
12	.5361	.3969	32	.4986	.5503
13	.5167	.4027	33	.5034	.5529
14	.50[illegible]	.4111	34	.5078	.5550
15	.4865	.4211	35	.5120	.5567
16	.4756	.4320	36	.5164	.5580
17	.4675	.4432	37	.5202	.5590
18	.4615	.4544	38	.5233	.5597
19	.4578	.4652	39	.5270	.5600
20	.4560	.4756	40	.5295	.5601

Note: The calculation for Case 3 was also repeated with $N = 99$, and values of $|\lambda_n^{\gamma_2} c_n|$ showed similar behaviour, e.g.

n	20	30	40	50	60	70	80	90	99
$\lvert\lambda_n^{\gamma_2} c_n\rvert \times 10^{-2}$	.10192	.12137	.12899	.13002	.12777	.12407	.11998	.11611	.11308

with smooth variation between these values. The values differ from those in the table by the factor $|\lambda_n^{\gamma_2}|/n^{2.74}$ which $\sim \pi^{2.74} = 23.0245$ as $n \to \infty$.

(ii) Data not satisfying the edge condition $f^{(1)}(1) = 0$

To investigate the effect of data that violate the compatability condition $\Psi_{xy}(0,1) = 0$, it is sufficient to consider the case

$$f^{(1)}(y) = y, \quad f^{(2)}(y) = 0 \tag{4.24}$$

since smooth data of the type already considered can be superposed on this to produce a general distribution. Referred to the corner (0,-1), $f^{(1)} = -1 + Y$ so that

$$\hat{f}^{(1)} = -\frac{1}{s+1} + \frac{1}{s+2} + \ldots$$

In this case from (4.7) with $\hat{h} = \hat{f}^{(1)}$, $\hat{g} = 0$, we have

$$\hat{p} = -\left(\frac{\sin\pi s}{\Delta}\right)\hat{f}^{(1)}$$

which has a simple pole at $s = -1$ with residue $\frac{\pi}{2}$ and is regular at $s = -2$, the next poles being those at $-\gamma_2$, $-\bar{\gamma}_2$.

Therefore

$$p = \frac{\pi}{2} + O\left(Y^{\gamma_2 - 1}\right),$$

and leading term in c_n is obtained by writing $f^{(4)} = \frac{\pi}{2}$, $f^{(2)} = 0$ in (2.12). This gives

:

$$c_n \sim \frac{1}{2} \cdot \frac{\pi}{2}\left\langle -\lambda_n^2 \psi_n \right\rangle = \frac{\pi}{2}\left(\frac{\tan\lambda_n}{\lambda_n}\right) \tag{4.25}$$

the next terms being $O(\lambda_n^{-\gamma_2})$.

Since $\tan \lambda_n = i - \frac{1}{2\lambda_n} + O(\lambda_n^{-2})$, this result shows that

$$|c_n \lambda_n| \rightarrow \frac{\pi}{2} \quad \text{as } n \rightarrow \infty \tag{4.26}$$

which is well borne out by a calculation using the method of optimal weighting functions, as shown in table 4. Except for $n = 1$, the computed value of $|\lambda_n c_n|$ lies within 2 percent of $\frac{\pi}{2} = 1.5708$, and the coefficients in fact approach the pure imaginary form $c_n \sim \frac{i}{2n}$ implied by (4.25)

Table 4 Data violating the compatibility condition

$$f^{(1)} = y,\ f^{(2)} = 0 \qquad d_m = -\frac{1}{\lambda_m} + \frac{3\tan\lambda_m}{\lambda_m^2}$$

The coefficients were computed with $N = 99$.

n	a_n	b_n	$\lvert\lambda_n c_n\rvert$
1	0.243986,+00	0.777206,+00	0.194527,+01
2	0.376416,-01	0.277808,+00	0.156334,+01
3	0.213641,-01	0.174970,+00	0.153696,+01
4	0.142829,-01	0.129271,+00	0.154212,+01
5	0.101696,-01	0.102807,+00	0.154905,+01
6	0.755116,-02	0.853834,-01	0.155421,+01
7	0.578495,-02	0.730035,-01	0.155763,+01
8	0.453904,-02	0.637432,-01	0.155979,+01
9	0.362747,-02	0.565529,-01	0.156105,+01
10	0.294002,-02	0.508079,-01	0.156168,+01
11	0.240808,-02	0.461126,-01	0.156188,+01
12	0.198738,-02	0.422037,-01	0.156176,+01
13	0.164834,-02	0.388992,-01	0.156142,+01
14	0.137055,-02	0.360694,-01	0.156091,+01
15	0.113968,-02	0.336189,-01	0.156027,+01
16	0.945324,-03	0.314764,-01	0.155954,+01
17	0.779874,-03	0.295875,-01	0.155874,+01
18	0.637593,-03	0.279097,-01	0.155789,+01
19	0.514141,-03	0.264095,-01	0.155700,+01
20	0.406145,-03	0.250602,-01	0.155608,+01
21	0.310976,-03	0.238402,-01	0.155514,+01
22	0.226564,-03	0.227318,-01	0.155418,+01
23	0.151235,-03	0.217204,-01	0.155320,+01
24	0.836376,-04	0.207937,-01	0.155223,+01
25	0.226839,-04	0.199417,-01	0.155124,+01
26	-0.325382,-04	0.191555,-01	0.155025,+01
27	-0.827604,-04	0.184280,-01	0.154927,+01
28	-0.128630,-03	0.177527,-01	0.154828,+01
29	-0.170660,-03	0.171243,-01	0.154730,+01
30	-0.209291,-03	0.165381,-01	0.154633,+01
31	-0.244919,-03	0.159899,-01	0.154536,+01
32	-0.277862,-03	0.154762,-01	0.154439,+01
33	-0.308389,-03	0.149938,-01	0.154344,+01
34	-0.336752,-03	0.145400,-01	0.154249,+01
35	-0.363157,-03	0.141122,-01	0.154155,+01
36	-0.387793,-03	0.137084,-01	0.154062,+01
37	-0.410818,-03	0.133266,-01	0.153970,+01
38	-0.432371,-03	0.129649,-01	0.153878,+01
39	-0.452582,-03	0.126220,-01	0.153788,+01
40	-0.471558,-03	0.122963,-01	0.153699,+01

(iii) Data containing discontinuities

We next consider the case when one of the data functions $f^{(1)}$, $f^{(2)}$ has a finite discontinuity at an internal point $y = c$ of $(0,1)$, and is elsewhere continuously differentiable. In this case the solution sufficiently close to the point c is in the limit the same as in a half space $x > 0$. For the half space, the following singular integral relation exists between the second derivatives at the surface $x = 0$ of a function Ψ satisfying the biharmonic equation (appendix E)

$$\Psi_{xx}(0,y) - \Psi_{yy}(0,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\Psi_{xy}(0,t)dt}{t-y} , \tag{4.27}$$

the integral on the right being a Cauchy principal value. The integral is continuous at a point of continuity of $\Psi_{xy}(0,t)$, and is continuously differentiable if $\Psi_{xy}(0,t)$ is so. Therefore a point of continuity of the boundary value of Ψ_{xy} is also a point of continuity of $\Psi_{xx} - \Psi_{yy}$, and if Ψ_{yy} has a jump discontinuity at a point $y = c$ of the boundary, Ψ_{xx} has an equal jump at the same point, and $\Delta\Psi$ a jump of twice the amount.

Thus for the strip, if $f^{(1)}$ is continuous at c while $f^{(2)}$ has a discontinuity of amount

$$f^{(2)}(c+) - f^{(2)}(c-) \equiv [f^{(2)}]_c \quad \text{say} \tag{4.28a}$$

there will also be a discontinuity in $f^{(4)}$ of amount

$$f^{(4)}(c+) - f^{(4)}(c-) = 2\,[f^{(2)}]_c \tag{4.28b}$$

The contribution to c_n is therefore given by (2.12) as

$$\frac{1}{2}\int_0^c \left\{ f^{(2)}(c-)\,\phi_n^{(4)} + f^{(4)}(c-)(\phi_n^{(2)} - \phi_n^{(4)}) \right\} dy \qquad (4.29)$$

$$+ \frac{1}{2}\int_c^1 \left\{ f^{(2)}(c+)\,\phi_n^{(4)} + f^{(4)}(c+)(\phi_n^{(2)} - \phi_n^{(4)}) \right\} dy$$

The dominant contribution is that from $\phi_n^{(2)} \equiv \psi_n''$, and is

$$\left(\frac{-1}{2}\right) [f^{(4)}(c+) - f^{(4)}(c-)]\ \psi_n'(c)$$

$$= -\ [f^{(2)}]_c \left(\frac{\sin\lambda_n c}{\sin\lambda_n} - c\ \frac{\cos\lambda_n c}{\cos\lambda_n} \right)$$

$$\sim -\ [f^{(2)}]_c \quad (1-c) \quad e^{i\lambda_n(1-c)}$$

Since $\mathrm{Im}\ \lambda_n \sim \frac{1}{2}\ \ln 4n\pi$, we obtain from this expression the estimate

$$|c_n| \ \sim\ (1-c)\ |[f^{(2)}]_c|\,(4n\pi)^{-\left(\frac{1-c}{2}\right)} \qquad (4.30)$$

In [1], the coefficients generated by the distribution

$$f^{(1)} = 0, \qquad f^{(2)} = \begin{cases} 1 & 0 \leq y < \frac{1}{2} \\ -1 & \frac{1}{2} < y \leq 1 \end{cases} \qquad (4.31)$$

were calculated using the method of optimal weighting functions. This falls into the pattern just discussed, with $c = \frac{1}{2}$, $[f^{(2)}]_c = -2$. The estimate for this case is therefore

$$n^{1/4}\ |c_n| \to (4\pi)^{-1/4} = 0.5311 \qquad (4.32)$$

This is borne out very well by the calculated values, as shown in Table 5 . Alternate values are above and below .5311, the means of successive pairs of values showing rapid convergence towards this value.

Case 2: $f^{(1)} = 0, \qquad f^{(2)} = \begin{cases} 1 & y \in (0,\frac{1}{2}) \\ -1 & y \in (\frac{1}{2},1) \end{cases}$

For this case $d_m = -\dfrac{\tan\lambda_m}{\lambda_m} + 2\psi_m'(\tfrac{1}{2}) + 2\,\dfrac{\sin\left(\frac{\lambda_m}{2}\right)}{\lambda_m \cos\lambda_m}$.

(The last two terms can be combined as

$$\left(\frac{\tan\lambda_m}{\lambda_m}\right)\left|-\frac{1}{2}\cos\left(\frac{3\lambda_m}{2}\right) + \frac{3}{2}\cos\left(\frac{\lambda_m}{2}\right)\right| \quad ,$$

the form given in [1]).
The coefficients $c_n = a_n + ib_n$ computed with $N = 40$ are listed on page 40 reference 1.

n	$n^{1/4}\lvert c_n\rvert$	n	$n^{1/4}\lvert c_n\rvert$	Mean of successive values
1	.86252	21	.60434	
2	.49262	22	.47357	.5389
3	.73997	23	.60258	.5380
4	.46516	24	.47255	.5376
5	.70536	25	.59771	.5351
6	.43944	26	.47743	.5376
7	.56952	27	.59695	.5372
8	.45011	28	.47812	.5365
9	.65088	29	.59260	.5354
10	.45364	30	.48059	.5366
11	.63447	31	.59243	.5365
12	.45633	32	.47912	.5358
13	.62715	33	.58852	.5338
14	.46245	34	.48325	.5359
15	.61982	35	.58870	.5360
16	.46286	36	.48168	.5352
17	.61347	37	.58516	.5334
18	.46875	38	.48552	.5353
19	.60988	39	.58556	.5355
20	.46821	40	.48389	.5347

$(4\pi)^{-1/4} = 0.531126$

(iv) Discontinuity in a derivative

Continuing the study of discontinuous data, suppose next that $f^{(2)}$ is continuous, but has a jump in first derivative, of amount

$$f^{(2)'}(c+) - f^{(2)'}(c-) = [f^{(2)'}]_c \tag{4.33a}$$

at a point $c \in (0,1)$. Then from the last section it follows that provided $f^{(1)}$ is continuously differentiable at c, there is a corresponding jump in $f^{(4)'}$, of amount

$$f^{(4)'}(c+) - f^{(4)'}(c-) = 2\,[f^{(2)'}]_c \tag{4.33b}$$

The expression (2.12) for c_n, after one integration by parts, gives

$$c_n = \left(-\frac{1}{2}\right)\left\langle \left(f^{(2)'}(y) - f^{(4)'}(y)\right)\left(-\frac{2\sin\lambda y}{\lambda\cos\lambda}\right) + f^{(4)'}(y)\psi'(y)\right\rangle_{\lambda=\lambda_n}$$

(where we have also used the fact that, for compatible data, $f^{(4)}(1) = f^{(2)}(1)$)

The dominant term is the last in the bracket, which contributes

$$c_n \sim \left(\frac{1}{2}\right)\left(f^{(4)'}(c+) - f^{(4)'}(c-)\right)\psi_n(c) \tag{4.34}$$

As before, asymptotic results show that

$$\psi_n(c) \sim \left(\frac{i}{\lambda_n}\right)(1-c)e^{i\lambda_n(1-c)}, \qquad \lambda_n \sim n\pi + \frac{i}{2}\ln 4n\pi$$

whence

$$|c_n| \sim \left(\frac{1-c}{n\pi}\right)(4n\pi)^{-\left(\frac{1-c}{2}\right)}\,\left|[f^{(2)'}]_c\right| \tag{4.35}$$

The coefficients generated by the distribution

$$f^{(1)} = 0, \qquad f^{(2)} = \begin{cases} 1 & 0 \leq y < \tfrac{1}{2} \\ 1 - 8(y-\tfrac{1}{2}) & \tfrac{1}{2} < y \leq 1 \end{cases} \tag{4.36}$$

have also been calculated by the method of optimal weighting functions, and are listed in Table 6. For this case $c = \frac{1}{2}$, $[f^{(2)\prime}]_c = -8$ so our estimate is

$$|c_n| \sim \frac{2^{3/2}}{(n\pi)^{5/4}} = .67625n^{-5/4} \tag{4.37}$$

and this is borne out by the computations, again with oscillations above and below in successive terms.

The right hand side, calculated from (4.36), is

$$d_m = \left\langle \left[\psi''_m + \frac{\cos\lambda_m y}{\cos\lambda_m}\right] f^{(2)} \right\rangle \quad (\text{since } f^{(1)} = 0)$$

$$= -\frac{3\tan\lambda_m}{\lambda_m} - 4\,\frac{\sin\left(\frac{\lambda_m}{2}\right)}{\lambda_m\cos\lambda_m}\left(1 + \frac{2}{\cos\lambda_m}\right)$$

$$-\frac{8}{\lambda_m^2}\left[1 - \frac{\cos\left(\frac{\lambda_m}{2}\right)}{\cos\lambda_m}\right] \tag{4.38}$$

The second term here equals $-8\psi(\frac{1}{2})$, and has precisely the same asymptotic behaviour as (4.37).

Table 6 Computations made using the method of optimal weighting functions

RHS is (0, 1 0 < Y < 0.5; 5 - 8y 0.5 < Y < 1)

Number of equations is 50

		a_n	b_n	$n^{5/4}\|c_n\|$	Mean of successive values
C	1	0.85217905,+00	0.82889897,+00	1.18889	
C	2	0.15110750,+00	0.31273898,+00	.82610	
C	3	-0.23383817,+00	-0.92803374,-01	.99330	
C	4	-0.11630692,+00	-0.17379686,+00	1.18298	
C	5	0.81507437,-01	-0.16184736,-01	.62131	
C	6	0.53842954,-01	0.69301344,-01	.82411	
C	7	-0.52680854,-01	-0.26550267,-02	.60059	
C	8	-0.33598252,-01	-0.57684090,-01	.89815	
C	9	0.39962947,-01	-0.71828929,-02	.63294	
C	10	0.26417607,-01	0.38246136,-01	.82660	
C	11	-0.28607324,-01	0.36837519,-02	.57782	
C	12	-0.18216213,-01	-0.31931769,-01	.82107	
C	13	0.24928523,-01	-0.48384923,-02	.62684	
C	14	0.16112214,-01	0.25319424,-01	.81273	
C	15	-0.19434066,-01	0.40120482,-02	.58579	
C	16	-0.12109066,-01	-0.21474948,-01	.78892	
C	17	0.17645042,-01	-0.37712050,-02	.62285	
C	18	0.11117389,-01	0.18456077,-01	.79883	
C	19	-0.14567490,-01	0.35850678,-02	.59511	
C	20	-0.88840248,-02	-0.15939817,-01	.77181	
C	21	0.13460482,-01	-0.31213155,-02	.62117	.69649
C	22	0.82831224,-02	0.14294083,-01	.78714	.70416
C	23	-0.11554256,-01	0.31138903,-02	.60274	.69494
C	24	-0.69142859,-02	-0.12550230,-01	.76116	.68195
C	25	0.10782529,-01	-0.26671130,-02	.62093	.69105
C	26	0.64976760,-02	0.11540997,-01	.77759	.69926
C	27	-0.95120892,-02	0.27102167,-02	.60874	.69317
C	28	-0.55983579,-02	-0.10276338,-01	.75374	.68124
C	29	0.89361062,-02	-0.23263043,-02	.62142	.68758
C	30	0.52875285,-02	0.96042473,-02	.76976	.69559
C	31	-0.80422256,-02	0.23802368,-02	.61350	.69163
C	32	-0.46653040,-02	-0.86527597,-02	.74818	.68084
C	33	0.75939475,-02	-0.20592124,-02	.62232	.68525
C	34	0.44224351,-02	0.81778625,-02	.76330	.69281
C	35	-0.69375644,-02	0.21117078,-02	.61735	.69033
C	36	-0.39725085,-02	-0.74400338,-02	.74374	.68055
C	37	0.65785572,-02	-0.18437055,-02	.62345	.68360
C	38	0.37777461,-02	0.70893951,-02	.75790	.69068
C	39	-0.60795518,-02	0.18915114,-02	.62053	.68922
C	40	-0.34414227,-02	-0.65025212,-02	.74008	.68031
C	41	0.57856459,-02	-0.16661447,-02	.62464	.68236
C	42	0.32814420,-02	0.62349685,-02	.75334	.68899
C	43	-0.53957403,-02	0.17088882,-02	.62322	.68828
C	44	-0.30225881,-02	-0.57580215,-02	.73695	.68009
C	45	0.51511093,-02	-0.15172823,-02	.62587	.68141
C	46	0.28890434,-02	0.55486970,-02	.74942	.68765
C	47	-0.48390226,-02	0.15556802,-02	.62551	.68747
C	48	-0.26852842,-02	-0.51537342,-02	.73422	.67987
C	49	0.46326793,-02	-0.13908531,-02	.62707	.68065
C	50	0.25724987,-02	0.49868710,-02	.74606	.68657

$(2^{3/2}\pi^{-5/4} = .67625)$

5. Convergence of the derived expansions

The estimates obtained in section 4 for rates of convergence of the $\{c_n\}$ throw light on the convergence of the series

$$\sum c_n \phi_n^{(1)}(z), \qquad \sum c_n \phi_n^{(2)}(y) \tag{5.1}$$

for the same four types of data :

(i) Smooth data satisfying the compatability conditions

We recall from Appendix B that the L_2 norms

$$\| \phi_n^{(1)} \|, \quad \| \phi_n^{(2)} \| \text{ are of order } n/(\ln n)^{3/2} \tag{5.2}$$

while the L_∞ norms are

$$\sup_y |\phi_n^{(1)}(y)|, \quad |\phi_n^{(2)}(y)| \quad \sim \quad 0\,(n/\ln n) \tag{5.3}$$

In this case, by (4.22), $|c_n| \sim n^{-2.74}$, so the series (5.1) are absolutely and uniformly convergent. The fact that they converge to the prescribed data functions $f^{(1)}$, $f^{(2)}$ is proved by the author in another paper [ ]. Gregory (1980) has given a proof of completeness of the functions $\{\phi_n^{(1)}, \phi_n^{(2)}\}$ for twice differentiable $f^{(1)}$, $f^{(2)}$, with $f^{(1)''}$, $f^{(2)''}$ of bounded variation on [0,1].

(ii) Data violating compatability

In this case, from (4.26), $|c_n| = 0(\frac{1}{n})$, and it is not possible to guarantee that the series (5.1) are convergent. It is in any case impossible for $\sum c_n \phi_n^{(1)}(y)$ to converge at $y = 1$ to a sum different from zero, since each $\phi_n^{(1)}(1) = 0$, whereas for the distribution examined in section 4(ii), $f^{(1)}(1) = 1$,

and convergence of the series near $y = 1$ could not be expected. In fact the partial sums were found to be oscillatory for fixed y, but the work of Joseph & Sturges (1978) suggests that the series should be Césaro summable to the data as all points except $y = 1$. This was confirmed in calculations by Mayes (1982) for the corresponding problem for a cylinder.

However the individual terms of the series can be integrated to give

$$\int_0^y \phi_n^{(1)}(t)\,dt = -\lambda_n\,(\psi_n(y) - \psi_n(0)) \tag{5.4}$$

$$\int_0^y \phi_n^{(2)}(t) = -\,\phi_n^{(1)}(y)/\lambda_n \tag{5.5}$$

The norms of these integrals are asymptotically of order $\frac{1}{n}$ times the corresponding norms (5.2), (5.3).

Therefore for $|c_n|$ of order $\frac{1}{n}$, the series

$$\sum c_n \int_0^y \phi_n^{(1)}(t)\,dt, \qquad \sum c_n \int_0^y \phi_n^{(2)}(t)\,dt \tag{5.6}$$

are certainly summable in L_2 norm, and computations in Table 7a show that with good accuracy these series sum pointwise to

$$\int_0^y f^{(1)}(t)dt\ (= \tfrac{1}{2}y^2), \qquad \int_0^y f^{(2)}(t)dt\ (= 0) \tag{5.7}$$

respectively. The quantity σ quoted in the table is the least squares error given in each case by

$$\sigma^2 = \frac{1}{k}\sum_{i=1}^{k}\left(\int_0^{y_i} f(t)dt - S(y_i)\right)^2 \tag{5.8}$$

where S is the relevant sum (5.6) evaluated at $y_i = \frac{i}{k}$ $(k = 10)$.

(iii) Discontinuity in $f^{(2)}$

From (4.30) in this case $|c_n| = 0\left(n^{-\left(\frac{1-c}{2}\right)}\right)$

Therefore $|c_n| \| \phi_n^{(1)} \|$, $|c_n| \| \phi_n^{(2)} \|$ are of order $n^{\left(\frac{1+c}{2}\right)} (\ln n)^{-3/2}$, and the series is not summable. However termwise integration twice with respect to y introduces a factor n^{-2}, so that the series

$$\sum c_n \int_0^y dt \int_0^t \phi_n^{(\alpha)}(u)\, du, \qquad \alpha = 1, 2 \tag{5.9}$$

are absolutely and uniformly summable. For the case computed, with $c = \frac{1}{2}$, the calculations summerised in table 7b show that the sums agree with the twice-integrated data with error of order 10^{-3}.

(iv) Discontinuity in $f^{(2)'}$

Similar remarks apply in this case, but since $|c_n| = 0\left(n^{-\frac{3-c}{2}}\right)$, it is only necessary to integrate once. The case $c = \frac{1}{2}$ gives rise to the figures in table 7c showing comparable agreement with the once-integrated data.

Table 7 Convergence of series of integrated terms

[$\int$ denotes $\int_0^y dy$ in all cases]

7a: Data violating compatability condition

$f^{(1)} = y, \quad f^{(2)} = 0 \qquad c_n \sim \frac{\pi}{2} \frac{\tan\lambda_n}{\lambda_n} \sim \frac{i}{2n}$

y	$\int f^{(1)} = \frac{1}{2}y^2$	$\sum c_n \int \phi_n^{(1)}$	$\sum c_n \int \phi_n^{(2)}$
0	0	0	0
.1	.005	.00377	-.00008
.2	.020	.02004	.00016
.3	.045	.04370	-.00027
.4	.080	.08016	.00038
.5	.125	.12356	-.00057
.6	.180	.18035	.00078
.7	.245	.24340	-.00112
.8	.320	.32055	.00153
.9	.405	.40336	-.00218
1.0	.500	.49934	0
		σ =.00102	σ = .00093

7b: Discontinuity in function

$f^{(1)} = 0, \quad f^{(2)} = \begin{cases} 1 \\ -1 \end{cases} \qquad |c_n| \sim n^{-1/4}$

y	$\iint f^{(2)}$	$\sum c_n \iint \phi_n^{(2)}$	$\sum c_n \iint \phi_n^{(1)}$
0	0	0	0
.1	.005	.00510	.00037
.2	.02	.01999	.00076
.3	.045	.04511	.00111
.4	.08	.07997	.00153
.5	.125	.12514	.00186
.6	.170	.16995	.00229
.7	.205	.20517	.00260
.8	.230	.22993	.00304
.9	.245	.24515	.00336
1.0	.250	.25007	.00377
		(σ = .00010)	(σ = .00223)

7c: **Discontinuity in first derivative**

$f^{(1)} = 0$ $\quad$ $f^{(2)} = 1 - 8\,[y-\tfrac{1}{2}]$ $\quad$ $|c_n| \sim n^{-5/4}$

y	$f^{(2)}$	$\sum c_n \phi_n^{(2)}$	$f^{(2)}$	$\sum c_n \phi_n^{(2)}$	
0	1	-4.463	0	0	
.1	1	-4.726	.1	.09994	$\sum c_n \phi_n^{(1)}$
.2	1	-5.464	.2	.20013	differs
.3	1	-6.807	.3	.29980	from zero
.4	1	-8.310	.4	.40027	with
.5	1	-5.999	.5	.49963	σ= .00062
.6	.2	-.996	.56	.56046	
.7	-.6	-1.023	.54	.53947	
.8	-1.4	-2.638	.44	.44055	
.9	-2.2	-5.283	.26	.25956	
1.0	-3	-3.742	0	0	

(σ = .00034)

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Appendix A

1. Stresses and displacements in terms of Airy stress function

With standard suffix notation the stresses are

$$\sigma_{11} = \Psi_{,22}, \quad \sigma_{12} = -\Psi_{,12}, \qquad \sigma_{22} = \Psi_{,11} \tag{A1}$$

The displacement gradients in plane strain are given by

$$\left.\begin{aligned} 2\mu u_{1,1} &= (1-\nu)\sigma_{11} - \nu\sigma_{22}, \quad 2\mu u_{2,2} = -\nu\sigma_{11} + (1-\nu)\sigma_{22}, \\ \mu(u_{1,2} &+ \mu_{2,1}) = \sigma_{12} \end{aligned}\right\} \tag{A2}$$

To treat boundary value problems it is convenient to introduce $P = \Delta\Psi$, together with its harmonic conjugate Q linked through the Cauchy-Riemann equations $P_{,1} = Q_{,2}$, $P_{,2} = -Q_{,1}$ and defined so Q (0,0) = 0.

The direct strains are then

$$2\mu\, u_{1,1} = \Psi_{,22} - \nu P, \quad 2\mu\, u_{2,2} = \Psi_{,11} - \nu P = -\Psi_{,22} + (1-\nu)P \tag{A3}$$

while the shear strains are

$$\left.\begin{aligned} 2\mu\, u_{1,2} &= -\Psi_{,12} - (1-\nu)Q \\ 2\mu\, u_{2,1} &= -\Psi_{,12} + (1-\nu)Q \end{aligned}\right\} \tag{A4}$$

Thus if boundary values are defined on $x_1 = 0$ as

$$\begin{pmatrix} f^{(1)} \\ f^{(2)} \\ f^{(3)} \\ f^{(4)} \end{pmatrix} = \begin{pmatrix} \Psi_{,12} \\ \Psi_{,22} \\ Q \\ P \end{pmatrix}_{x_1 = 0} \tag{A5}$$

the boundary values of the tractions and of the displacement gradients with respect to x_2 are respectively

$$\sigma_{12} = - f^{(1)} \qquad \sigma_{11} = f^{(2)}, \tag{A6}$$

$$2\mu \frac{\partial}{\partial x_2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} - f^{(1)} - (1-\nu) f^{(3)} \\ - f^{(2)} + (1-\nu) f^{(4)} \end{bmatrix} \tag{A7}$$

Therefore

(i) if $u_1\ (0,x_2)$ and $\sigma_{12}\ (0,x_2)$ are prescribed, $f^{(1)}$ and $f^{(3)}$ are known

(ii) if $u_2(0,x_2)$ and $\sigma_{11}(0,x_2)$ are prescribed, $f^{(2)}$ and $f^{(4)}$ are known.

(iii) (the stress problem) : if $\sigma_{12}(0,x_2)$ and $\sigma_{11}(0,x_2)$ are prescribed, $f^{(1)}$ and $f^{(2)}$ are known

(iv) (the displacement problem) : if $u_1(0, x_2)$ and $u_2\ (0,x_2)$ are prescribed, the combinations

$$\left. \begin{aligned} g^{(1)} &= f^{(1)} + (1-\nu)\ f^{(3)} \\ \text{and} \qquad g^{(2)} &= f^{(2)} - (1-\nu)\ f^{(4)} \end{aligned} \right\} \tag{A8}$$

are known.

(i) and (ii) are canonical problems, (iii) and (iv) are non-canonical.

2. Steady Stokes flow

Neglecting inertia terms, the equation of motion is taken as

$$\nabla p = \mu \Delta \underset{\sim}{q} \tag{A9}$$

where p is pressure and $\underset{\sim}{q} = (u_1, u_2)$ is the velocity, expressible in terms of

the stream function as $u_1 = \Psi_{,2}$, $u_2 = -\Psi_{,1}$.

Then with P,Q defined as in the previous subsection, (A9) may be integrated to

$$p = -\mu Q + \text{constant} \tag{A10}$$

The stresses are

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) - p\,\delta_{ij}$$

whence

$$\frac{1}{2\mu} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} \Psi_{,12} + \tfrac{1}{2}Q - p_o/2\mu \\ \Psi_{,22} - \tfrac{1}{2}P \end{bmatrix} \tag{A11}$$

Hence corresponding to the boundary value problems for the elastic strip, those for Stokes flow in a semi-infinite trench are

(i) u_2 and σ_{11} prescribed on $x_1 = 0 \Rightarrow f^{(1)}, f^{(3)}$ given

(ii) u_1 and σ_{12} prescribed on $x_1 = 0 \Rightarrow f^{(2)}, f^{(4)}$ given

(iii) u_1 and u_2 prescribed " " $\Rightarrow f^{(1)}, f^{(2)}$ given

(iv) σ_{11} and σ_{12} prescribed " " $\Rightarrow g^{(1)}, g^{(2)}$ given,

where $g^{(1)} = f^{(1)} + \tfrac{1}{2} f^{(3)}$, $g^{(2)} = f^{(2)} - \tfrac{1}{2} f^{(4)}$ (A12)

This is the same combination of data as arises in the case of the elastic displacement problem (A8) with $\nu = \tfrac{1}{2}$.

Appendix B Quadratures and Norms of eigenfunctions

The quadratures quoted in section 3 are derived from the following results, all of which are obtained by integration by parts and use of the equation and boundary conditions satisfied by the ψ_k [i.e. $(D^2 + \lambda_k^2)^2\psi_k = 0$, $\psi_k(1) = D\psi_k(1) = 0$, $D\psi_k(0) = D^3\psi_k(0) = 0$, together with the characteristic equation $2\lambda_k + \sin 2\lambda_k = 0$].

1 Quadratures

(a) $m \neq n$ All results expressible in terms of

$$C_{mn} = (\cos\lambda_m \cos\lambda_n)^{-1} \left\langle \cos\lambda_m y \cos\lambda_n y \right\rangle = (\lambda_m \tan\lambda_m - \lambda_n \tan\lambda_n)/(\lambda_m^2 - \lambda_n^2)$$

We find (more details are given in Appendix A of [1]) :

$$\left\langle \phi_m^{(1)} \phi_n^{(1)} \right\rangle = -4\lambda_m\lambda_n (\lambda_m^2 + \lambda_n^2) C_{mn} / (\lambda_m^2 - \lambda_n^2)^2$$

$$\left\langle \phi_m^{(2)} \phi_n^{(2)} \right\rangle = -8\lambda_m^2 \lambda_n^2 C_{mn} / (\lambda_m^2 - \lambda_n^2)^2$$

$$\left\langle \phi_m^{(1)} \phi_n^{(3)} \right\rangle = -4\lambda_m \lambda_n C_{mn} / (\lambda_m^2 - \lambda_n^2)$$

$$\left\langle \phi_m^{(2)} \phi_n^{(4)} \right\rangle = -4 \lambda_n^2 C_{mn} / (\lambda_m^2 - \lambda_n^2)$$

$$\left\langle \phi_m^{(3)} \phi_n^{(3)} \right\rangle = 4 C_{mn} - 4 (\tan\lambda_m + \tan\lambda_n) / (\lambda_m + \lambda_n)$$

$$\left\langle \phi_m^{(4)} \phi_n^{(4)} \right\rangle = 4 C_{mn}$$

(b) $m = n$

$$\left\langle \phi_m^{(1)} \phi_m^{(1)} \right\rangle + \frac{1}{2} = \left\langle \phi_m^{(2)} \phi_m^{(2)} \right\rangle - \frac{1}{2} = 1 - \frac{1}{3} \left(\frac{\lambda_m}{\cos\lambda_m} \right)^2$$

$$\left\langle \phi_m^{(3)} \phi_m^{(3)} \right\rangle = 4/\cos^2 \lambda_m, \quad \left\langle \phi_m^{(4)} \phi_m^{(4)} \right\rangle = 0$$

$$\left\langle \phi_m^{(1)} \phi_m^{(3)} \right\rangle = \left\langle \phi_m^{(2)} \phi_m^{(4)} \right\rangle = 1.$$

The biorthogonality relations (2.9) and (2.10) follow at once from these results.

2. Norms

The norms are found from the preceding expressions by writing $n = -m$, so $\lambda_n = \bar{\lambda}_m$, $\phi_n^{(1)} = \overline{\phi_n^{(1)}}$ etc. The details are given in appendix B of [1], and only the asymptotic expressions for large m are quoted here (in fact they are close approximations for $m > 2$).

$$\| \psi_m \| \sim (2m\pi)^{-1} (\tfrac{1}{2}\ln 4m\pi)^{-3/2}$$

$$\| \phi_m^{(1)} \| , \| \phi_m^{(2)} \| \sim \tfrac{1}{2}m\pi/(\tfrac{1}{2} \ln 4m\pi)^{3/2}$$

$$\| \phi_m^{(3)} \| , \| \phi_m^{(4)} \| \sim 2/ (\ln 4m\pi)^{1/2}$$

Appendix C Solution of the biharmonic equation $\Delta^2\Psi = 0$ in the quarter-plane $0 \leq \theta \leq \frac{\pi}{2}$, with boundary conditions

$$\theta = 0 : \Psi_{xy} = 0, \qquad \Psi_{xx} = 0 \tag{C1}$$

$$\theta = \frac{\pi}{2} : \Psi_{xy} + (1-\nu)Q = g^{(1)}(y), \; \Psi_{yy} - (1-\nu)P = g^{(2)}(y) \tag{C2}$$

($g^{(1)}$, $g^{(2)}$ prescribed).

The biharmonic equation is automatically satisfied when Ψ has the form of a Mellin integral

$$\Psi = \frac{1}{2\pi i} \int r^{1-s} \hat{\Psi} \, ds \tag{C3}$$

with $\quad \hat{\Psi} = A\cos(s-1)\theta + B\sin(s-1)\theta + C\cos(s+1)\theta + D\sin(s+1)\theta$

From (C3) we have

$$P \equiv \Delta\Psi = \frac{1}{2\pi i} \int r^{-1-s} \hat{P} ds \tag{C4}$$

with conjugate

$$Q = \frac{1}{2\pi i} \int r^{-1-s} \hat{Q} ds \tag{C5}$$

$$\text{where} \quad \begin{pmatrix} \hat{P} \\ \hat{Q} \end{pmatrix} = (-4s) \begin{pmatrix} \cos(s+1)\theta & \sin(s+1)\theta \\ -\sin(s+1)\theta & \cos(s+1)\theta \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \tag{C6}$$

The boundary conditions (C 1) are applied in the form $\hat{\Psi} = 0$, $\frac{\partial\hat{\Psi}}{\partial\theta} = 0$ on $\theta = 0$, giving

$$\left.\begin{aligned} A + C &= 0 \\ (s-1)B + (s+1)D &= 0 \end{aligned}\right\} \tag{C7}$$

The derivatives on $\theta = \frac{\pi}{2}$ that occur in the boundary condition (C2),

namely

$$\Psi_{yy}\Big|_{x=0} = \Psi_{rr}\Big|_{\theta=\frac{\pi}{2}} , \qquad \Psi_{xy}\Big|_{x=0} \equiv - \frac{\partial}{\partial r}\left(\frac{1}{r}\ \frac{\partial \Psi}{\partial \theta}\right)\Big|_{\theta=\frac{\pi}{2}} \tag{C8}$$

are also expressible as Mellin integrals derived from (C3). When these results are combined, and A and B eliminated using (C7), the boundary data on $\theta = \frac{\pi}{2}$ are expressible in terms of the remaining constants C and D in the form

$$g^{(\alpha)}(y) = \frac{1}{2\pi i} \int y^{-1-s}\ \hat{g}^{(\alpha)} ds, \qquad \alpha = 1,\ 2 \tag{C9}$$

with $$\begin{pmatrix} \hat{g}^{(1)} \\ \hat{g}^{(2)} \end{pmatrix} = M(s) \begin{pmatrix} C \\ D \end{pmatrix} \tag{C10}$$

where M is the matrix $\begin{pmatrix} (s-2(1-\nu))\ \cos \frac{\pi s}{2} & (s-1+2\nu)\ \sin \frac{\pi s}{2} \\ (s+1-2\nu)\ \sin \frac{\pi s}{2} & -\ (s+2(1-\nu))\ \cos \frac{\pi s}{2} \end{pmatrix}$

The inverse matrix M^{-1} gives C and in terms of the $\hat{g}$'s, and the values of $\hat{P}$, $\hat{Q}$ on $\theta = \frac{\pi}{2}$ are therefore

$$\begin{pmatrix} \hat{P} \\ \hat{Q} \end{pmatrix}_{\theta=\frac{\pi}{2}} = 4s \begin{pmatrix} \sin \frac{\pi s}{2} & -\ \cos \frac{\pi s}{2} \\ \cos \frac{\pi s}{2} & \sin \frac{\pi s}{2} \end{pmatrix} M^{-1} \begin{pmatrix} \hat{g}^{(1)} \\ \hat{g}^{(2)} \end{pmatrix}$$

$$= \left(-\frac{2}{\Delta}\right) \begin{pmatrix} \sin \frac{\pi s}{2}\ \cos \frac{\pi s}{2} & s + \sin^2 \frac{\pi s}{2} - 2(1-\nu) \\ s - \sin^2 \frac{\pi s}{2} + 2\ (1-\nu) & \sin \frac{\pi s}{2}\ \cos \frac{\pi s}{2} \end{pmatrix} \begin{pmatrix} \hat{g}^{(1)} \\ \hat{g}^{(2)} \end{pmatrix}$$

(C11)

where $\Delta \equiv -\det M = s^2 + (3-4\nu)(\sin^2 \frac{\pi s}{2}) - 4(1-\nu)^2$ (C12)

The solution quoted in section 4 of the present paper is obtained by setting $\nu = 1$ in these results. Then

$$g^{(1)} = \Psi_{xy}\big|_{x=0} \quad , \quad g^{(2)} = \Psi_{yy}\big|_{x} = 0, \tag{C13}$$

and

$$\Delta = s^2 - \sin^2 \frac{\pi s}{2} . \tag{C14}$$

Appendix D: Asymptotic expressions for the integrals

$$\int_0^1 t^{\nu-1} \; e^{\pm i\lambda t} \, dt \tag{D1}$$

for $0 < \text{Re}\ \nu < 1, \quad \lambda \in \Lambda +.$

Here $\Lambda+$ is the set of zeros of $2\lambda + \sin 2\lambda$ lying in the right hand half plane so that, by (1.10), $\arg \lambda = 0\left(\frac{\ln n}{n}\right)$.

In each case the integrals are evaluated as the difference between two infinite integrals

$$\text{(i)} \quad \int_0^1 t^{\nu-1} \; e^{i\lambda t} \; dt = \int_0^{i\infty} - \int_1^{i\infty}$$

These integrals both exist since $\text{Im}\ \lambda > 0$. The first is

$\lambda^{-\nu} \, e^{\,i\nu\pi/2} \, \Gamma(\nu)$, while integration by parts gives

$$\int_1^{i\infty} t^{\nu-1} e^{i\lambda t} dt = -\frac{e^{i\lambda}}{\lambda} \left(1 + 0\left(\frac{1}{\lambda}\right)\right)$$

Since (as $e^{i\lambda} = 0\ (\lambda^{-\frac{1}{2}})$, the last term is of order $\lambda^{-5/2}$, and altogether

$$\int_0^1 t^{\nu-1} \; e^{i\lambda t} dt = \lambda^{-\nu} e^{\,i\nu\pi/2} \Gamma(\nu) + \frac{e^{i\lambda}}{i\lambda} + 0(\lambda^{-5/2}) \tag{D2}$$

$$\text{(ii) Likewise} \quad \int_0^1 t^{\nu-1} e^{-i\lambda t} dt = \int_0^{-i\infty} - \int_1^{-i\infty}$$

and in the same manner we obtain

$$\int_0^1 t^{\nu-1} e^{-i\lambda t} dt = \lambda^{-\nu} \, e^{-\,i\nu\pi/2} \, \Gamma(\nu) - \frac{e^{-i\lambda}}{i\lambda} + 0\ (\lambda^{-3/2}) \tag{D3}$$

The results quoted in (4.20) are obtained by addition and subtraction of (D2) and (D3).

Appendix E.: Derivation of equation (4.27)

A direct derivation is provided by writing the solution of the biharmonic equation in the half plane x > 0 as a Fourier integral

$$\Psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi y} \; \hat{\Psi} (x,\xi) d\xi$$

where

$$\left[\left(\frac{d}{dx}\right)^2 - \xi^2\right]^2 \hat{\Psi} = 0$$

The solution bounded as $x \to \infty$ is $\hat{\Psi} = (A+Bx)e^{-x|\xi|}$

Then on x = 0,

$$\Psi_{xx} - \Psi_{yy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2|\xi|(|\xi|A-B)e^{-i\xi y} d\xi$$

while $$\Psi_{xy}(0,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi(|\xi|A-B)e^{-i\xi t} d\xi$$

Now the Cauchy integral $\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i\xi t} dt}{t-y} = - i \; e^{-i\xi y}(\mathrm{sgn}\xi)$

Hence on reversing the order of integration,

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\Psi_{xy}(0,t)dt}{t - y} = \frac{1}{\pi}\int_{-\infty}^{\infty} \xi(\mathrm{sgn}\xi) \; (|\xi|A-B)e^{-i\xi y} d\xi$$

$$= \Psi_{xx}(0,y) - \Psi_{yy} (0,y)$$

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We consider boundary value problems for the biharmonic equation in the open rectangle $x > 0$, $-1 < y < 1$, with homogeneous boundary conditions on the free edges $y = \pm 1$, and data on the end $x = 0$ of a type arising both in elasticity and in Stokes flow of a viscous fluid, in which either two stresses or two displacements are prescribed. For such "non-canonical" data, coefficients in the eigenfunction expansion can be found only from the solution of infinite sets of linear equations, for which a variety of methods of formulation have been proposed.

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20. ABSTRACT - cont'd.

A drawback of existing methods has been that the resulting equations are unstable with respect to the order of truncation. It is clear from an examination of the spectrum of a typical matrix that ill-conditioning is to be expected. However, a search among a wider class of possible trial functions than hitherto for use in a Galerkin method based on the actual eigenfunctions has led to the choice of a unique set, here termed optimal weighting functions, for which the resulting infinite matrix is diagonally-dominated. This ensures the existence of an inverse, which can be approximated by solving a finite subset of the equations.

Computations for a number of representative cases, presented in full in an internal report (Spence 1978) are summarized here, with emphasis on the rates of decay of the coefficients $\{c_m\}$ in the eigenfunction expansion. Knowledge of these decay rates is essential for a discussion of convergence, parallel to that given by Joseph (1977a,b) and his co-workers for canonical problems.

Asymptotic estimates of the decay rates have also been obtained by use of the solution of the biharmonic equation in a quarter plane. It is found that (i) for smooth continuous data satisfying compatibility conditions at the corners, the decay rates guarantee pointwise convergence. Also examined are (ii) cases of data violating compatibility (iii) discontinuous data and (iv) discontinuities in derivatives of the data. In these cases sharp estimates of convergence rates are obtained, which guarantee that integrals of the series converge to integrals of the data. The computations show striking confirmation of the theoretical estimates.

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